Inventory Models (Stock Control)

Reference books:

Anderson, Sweeney and Williams “An Introduction to Management Science, quantitative approaches to decision making” 7th edition


T A Burley, G O’sullivan, “Operational Research”

L Lapin, “Quantitative Methods for business decisions with cases”, 5th edition

Lecture 1

Inventory, sometimes known as stock, refers to idle goods or materials that are held by an organisation for use sometime in the future. Items carried in inventory can be:

- raw materials
- purchased parts
- components
- subassemblies
- work-in-process
- finished goods
- and supplies

One main reason that organisations maintain inventory is that it is rarely possible to predict sale levels, production times, demand, usage needs exactly. Thus, inventory serves as a buffer against uncertain and fluctuating usage and keeps a supply of items available in case the item are needed by the organisation or its customers. While inventory serves an important and essential role, the expense associated with financing and maintaining inventories is a substantial part of the cost of doing business. In large organisations, the cost associated with inventory can run into millions of dollars.

To minimise the inventory cost and to guarantee a smooth operation of the organisation the same time, the manager needs to answer the following two important questions:

1) **How much** should be ordered when the inventory for the item is replenished?

2) **When** should the inventory for a given item be replenished?

The purpose of the discussion on inventory models is to show how quantitative models can assist in making the above decisions. We face three situations:

- Deterministic inventory models, where it is reasonable to assume that the rate of demand for the item is constant or nearly constant;
• Probabilistic inventory models, where the demand for item fluctuates and can be described in probabilistic terms;
• Just-in-time (JIT), a philosophy of material management and control, of which the primary objective is to eliminate all sources of waste, including unnecessary inventory.

1. A Generalised Inventory Model

It has been mentioned earlier that the ultimate objective of any inventory model is to answer the “how much to order” and “when to order” questions. The answer to the first question is the **order quantity**. It represents the optimum amount that should be ordered every time an order is placed and may vary with time depending on the situation under consideration. The answer to the second question depends on the type of the inventory system:

**periodic review** at equal time intervals - the time for acquiring a new order usually coincides with the beginning of each time interval;

**continuous review** - a **review point** is usually specified by the inventory level at which a new order must be placed.

The order quantity (how much) and reorder point (when) are normally determined by minimising the total inventory cost that can be expressed as a function of these two variables. The total inventory cost is generally composed of the following components:

$$\text{Total inventory cost} = \text{Purchasing cost} + \text{Setup cost} + \text{Holding cost} + \text{Shortage cost}$$

The **purchasing cost** becomes an important factor when the commodity unit price becomes dependent on the size of the order. This situation is normally expressed in terms of a **quantity discount** or a **price break**, where the unit price of the item decreases with the increase of the ordered quantity. The **setup cost** (also known as **ordering cost**) represents the fixed charge incurred when an order is placed. Thus, frequent smaller orders will result in a higher setup cost than less frequent larger orders. The **holding cost**, which represents the costs of carrying inventory in stock (e.g., interest on invested capital, storage, handling, depreciation, and maintenance), normally increases with the level of inventory. The **shortage cost**, is a penalty incurred when we run out of stock of a needed commodity. It generally includes costs due to loss of customer’s goodwill as well as potential loss in income.

2. Deterministic Models

2.1 Single-item static (EOQ) model

This is the best known and most fundamental inventory model, which is applicable when the demand for an item has constant, or nearly constant, rate and when the
entire quantity ordered arrives in inventory at one point in time (instantaneous replenishment). It assumes no shortage in this inventory model.

Different reference books may use different notations for the inventory cost parameters. We will use the following:

\[ D = \text{demand rate for an item (quantity/unit time)} \]
\[ y = \text{the order quantity} = \text{maximum inventory level} \]
\[ K = \text{the setup cost every time an order is placed} \]
\[ h = \text{the holding cost per item per unit time} \]
\[ \text{TC} = \text{the total cost per unit time} \]

From the definitions of \( D \) and \( y \), we know that the inventory level reaches zero from its maximum level in \( y/D \) time units (this being also known as the \textbf{cycle time}), and that the number of orders to be placed per unit time is \( D/y \). Figure 1 below illustrates the inventory pattern for the EOQ inventory model.

Generally, the total inventory cost is as follows:

\[
\text{Total inventory cost} = \text{Purchasing cost} + \text{Setup cost} + \text{Holding cost} + \text{Shortage cost}
\]

However, in the current situation the shortage cost is non-existent because it assumes there is no shortage at all. Moreover, the purchasing cost will be a mere constant since in this model we assume that the commodity price is independent of the size of the order. The total cost function is to be used mainly for finding the optimal inventory level so that total cost is minimal. A constant in such a function will not alter the optimal solution. Therefore, in the EOQ model, we express the total inventory cost as the sum of the setup cost and the holding cost, as follows:

\[
\text{Total inventory cost} = \text{Setup cost} + \text{Holding cost}
\]
The setup cost equals to the product of the number of orders per unit time and the setup cost per order. That is,

$$\text{setup cost} = (D/y)K = DK/y \quad \text{(cost/unit time)}$$

The holding cost can be calculated by multiplying the average inventory level to the holding cost per item per unit time. The average inventory level in the EOQ model is \(\frac{1}{2} y\). Therefore,

$$\text{holding cost} = \left(\frac{1}{2} y\right) h = \frac{1}{2} yh \quad \text{(cost/unit time)}$$

Hence, the total cost function is

$$TC = \frac{DK}{y} + \frac{1}{2} yh \quad \text{(cost/unit time)} \quad (1)$$

The “how-much-to-order” decision

Equation (1) is a function of the order quantity \(y\). The optimal value of \(y\), the answer to the “how-much-to-order” question, can be obtained by minimising \(TC\) with respect to \(y\) in the following procedure:

1) differentiate \(TC\) with respect to \(y\)

$$\frac{dTC}{dy} = -\frac{DK}{y^2} + \frac{h}{2}$$

2) let the above be zero

$$-\frac{DK}{y^2} + \frac{h}{2} = 0$$

3) solve the equation for \(y^*\), the optimal order quantity

$$y^* = \sqrt{\frac{2KD}{h}} \quad (2)$$

The order quantity is the economic order quantity, sometimes also known as Wilson’s economic lot size.

The corresponding cycle time is:

$$t_0^* = \frac{y^*}{D}$$

and the minimal total inventory cost is:

$$TC^* = \sqrt{2KDh}$$

The “when-to-order” decision
To answer this question we need first to introduce the concept of inventory position. The inventory position for an item is defined as the amount of the inventory on hand plus the amount of inventory on order. The “when-to-order” decision is expressed in terms of a reorder point - the inventory position at which a new order should be placed.

In practice, an order takes time to be filled. The time lag from the point at which an order is placed until the order is delivered is called the lead time, denoted by L. Figure 2 illustrates the situation where reordering occurs L time units before delivery is expected.

In general, the reordering point (R) can be calculated by the following equation:

\[ R = L \times D \]  

In equation (3), D is the demand per unit time, L is the lead time.

Note that the lead time L may be either longer or shorter than the cycle time \( t_0 \). If \( L < t_0 \), use L directly in equation (3); if, on the other hand, \( L > t_0 \), \( (L-n\cdot t_0) \) should be used in equation (3) instead of L where \( n \) is the largest integer not exceeding \( L/t_0 \).

Example:

The daily demand for a commodity is approximately 100 units. Every time an order is placed, a fixed cost \( K \) of £100 is incurred. The daily holding cost \( h \) per unit inventory is 2 Pence. If the lead time is 12 days, determine the economic order quantity and the reorder point.

Solve:

We know that

\[
\begin{align*}
K &= £100 \text{ per order} \\
D &= 100 \text{ units/day} \\
h &= £0.02 \text{ per unit per day} \\
L &= 12 \text{ days}
\end{align*}
\]

Using equation (1), the EOQ is
And the cycle time is

\[ t_0^* = \frac{y^*}{D} = \frac{1000}{100} = 10 \]

Because in this case \( L > t_0 \), the reorder point is calculated as

\[ R = (L-t_0) \times D = (12-10) \times 100 = 200 \text{ units} \]

The above calculation shows that it is most economical to order 1000 units of the commodity when the inventory level for this commodity reaches 200 units.

**Lecture 2**

2.2 Single item static (EOQ) model with price breaks

In the EOQ model, the purchasing cost per unit time is neglected in the analysis because it is constant and hence should not affect the level of inventory. It often happens however, the purchasing price per unit may depend on the size of the quantity purchased. This situation usually occurs in the form of discrete **price breaks** or **quantity discounts**. In such cases, the purchasing price should be considered in the inventory model.

Consider the inventory mode with instantaneous stock replenishment and no shortage. Assume the cost per unit is \( c_1 \) for \( y < q \) and \( c_2 \) for \( y \geq q \), where \( c_1 > c_2 \) and \( q \) is the quantity above which a price break is granted. The total cost per unit time will now include the purchasing cost in addition to the setup cost and the holding cost.

The total cost per unit time with price break is

\[
TC = \begin{cases} 
Dc_1 + KD/y + \frac{1}{2} yh & \text{for } y<q \\
Dc_2 + KD/y + \frac{1}{2} yh & \text{for } y\geq q
\end{cases}
\]

These two functions are shown graphically in Figure 3. Disregarding the effect of price breaks for the moment, we let \( y_m \) be the quantity at which the minimum values of \( TC_1 \) and \( TC_2 \) occur.
Thus

\[ y_m = \sqrt{\frac{2KD}{h}} \]

The cost functions \( TC_1 \) and \( TC_2 \) in Figure 4 reveal that the determination of the optimum order quantity \( y^* \) depends on where \( q \), the price break point, falls with respect to the three zones I, II, and III shown in the figure. These zones are defined by determining \( q_1 (> y_m) \) from the equation

\[ TC_1 (y_m) = TC_2 (q_1) \]

Since \( y_m \) is known, the solution will yield the value of \( q_1 \). The optimum order quantity \( y^* \) is, as illustrated in Figure 4,

\[
\begin{align*}
  y^* &= y_m & \text{if } 0 \leq q < y_m & \text{(Zone I)} \\
  y^* &= q & \text{if } y_m \leq q < q_1 & \text{(Zone II)} \\
  y^* &= y_m & \text{if } q \geq q_1 & \text{(Zone III)}
\end{align*}
\]

The procedure for determining \( y^* \) may thus be summarised as follows:

1. Determine \( y_m = \sqrt{(2KD/h)} \). If \( q < y_m \) (zone I), then \( y^* = y_m \), and the procedure ends. Otherwise,
2. Determine \( q_1 \) from equation \( TC_1 (y_m) = TC_2 (q_1) \) and decide whether \( q \) falls in zone II or zone III.

1) If \( y_m < q < q_1 \) (zone II), the \( y^* = q \);
2) if \( q \geq q_1 \) (zone III), the \( y^* = y_m \).

**Example:**

Consider the inventory model with the following information. \( K = £10 \), \( h = £1 \), \( D = 5 \) units, \( c_1 = £2 \), \( c_2 = £1 \), and \( q = 15 \) unit.

First, we need to compute the \( y_m \).

\[
y_m = \sqrt{2KD/h} = \sqrt{2 \times 10 \times 5/1} = 10 \text{ units}
\]

Since \( q > y_m \), it is necessary to check whether \( q \) is in zone II or zone III. The value of \( q_1 \) is computed from

\[
TC_1 (y_m) = TC_2 (q_1)
\]

namely,

\[
Dc_1 + KD/y_m + \frac{1}{2} hy_m = Dc_2 + KD/q_1 + \frac{1}{2} hq_1
\]

Substitution yields

\[
q_1^2 - 30q_1 + 100 = 0
\]

This is obviously of the type \( ax^2 + bx + c = 0 \) whose solutions are

\[
x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

So, the solutions for \( q_1 \) are found to be either \( q_1 = 26.18 \) units or \( q_1 = 3.82 \) units. By definition, \( q_1 \) must be larger than \( y_m \). Therefore, \( q_1 = 26.18 \) units.

Since \( y_m < q < q_1 \), \( q \) is in zone II. Therefore, the optimum order quantity \( y^* = q = 15 \) units.

The associated total cost per unit time is

\[
TC_2(y^*) = TC_2 (15) = Dc_2 + KD/15 + 15h/2 = £15.83/\text{day}.
\]

**Lecture 3**

2.3 EOQ model with planned shortages

A **shortage** is a demand that cannot be supplied. In many cases, shortages are undesirable and should be avoided if possible. However, in the cases where the value of a unit commodity is very high, shortages are then desirable and used commonly in practice. An example of this type of situation is a new-car dealer’s inventory. It is uncommon for a specific car you want to be in stock. What they do is ask you to specify your requirements for the car and place an order for you after you have showed your commitment (sizeable deposit for example).
In this model, we assume that
a. the shortage cost is small;
b. no demand is lost because of shortage as the customers will back order, i.e., accept delayed delivery (this type of shortage being known as **backorders**); and
c. the replenishment is instantaneous.

The EOQ inventory model with planned shortages is illustrated in Figure 5.

In Figure 5,
- $y =$ the total ordering quantity
- $b =$ the inventory shortage, or the backorder
- $t_1 =$ the time units in which the inventory reaches the zero level
  \[ t_1 = \frac{(y-b)}{D} \]
- $t_2 =$ the time period from stock-out to the arrival of a new order
  \[ t_2 = \frac{b}{D} \]
- $T =$ the inventory cycle, $T = \frac{y}{D}$

In the above, as in the EOQ model $D$ is the demand rate of the commodity.

In this case, the total inventory cost can be expresses as the sum of the holding cost, setup cost, and the shortage cost, i.e.,

\[ TC = \text{holding cost} + \text{setup cost} + \text{shortage cost} \]

**Holding cost**

In the EOQ model, it is assumed that the holding cost per item per unit time is $h$. The component of holding cost is the product of the average inventory level and $h$. The average inventory level is calculated as follows:

\[ \frac{1}{2} (y-b) t_1 + 0 t_2]/T = \frac{(y-b)^2}{2y} \]
Therefore, the holding cost is
\[ h(y-b)^2/(2y) \] \text{(cost/unit time)}

**Setup cost**

It is known from the EOQ model that the cost per order is \( K \). The setup cost equals to \( K \) multiplying the number of orders in the given period of time.

The number of orders can be calculated by dividing the demand by the ordering quantity, i.e., \( D/y \). The setup cost rate is \( K \) per order, hence the total setup cost is
\[ KD/y \] \text{(cost/unit time)}

**Shortage cost**

If we assume that the shortage penalty is \( p \) per item per unit time, then the shortage cost is
\[ p \times \text{average backorder level} \]

Since the average backorder level is
\[ (0 t_1 + \frac{1}{2} bt_2)/T = b^2/(2y) \]
the shortage cost is
\[ b^2p/(2y) \] \text{(cost /unit time)}

Therefore, the total cost is
\[ TC = h(y-b)^2/(2y) + KD/y + b^2p/(2y) \] \text{(cost/unit time)} \hspace{1cm} (5)

Minimise \( TC \) with respect to the order quantity and to the planned backorders yields the optimal values of order quantity \( y^* \) and backorder \( b^* \):

\[ y^* = \sqrt{\frac{2DK}{h} \cdot \frac{h + p}{p}} \] \hspace{1cm} (6)
\[ b^* = \frac{h}{h + p} \cdot y^* \] \hspace{1cm} (7)

**Proof**

\[ \frac{\partial (TC)}{\partial (b)} = -\frac{h}{2y} \cdot 2(y-b) + \frac{2bp}{2y} = \frac{p}{y} b - h + \frac{h}{y} b = 0 \]

That gives
Substituting $b^* = \frac{h}{h+p} y^*$ in the above equation gives

$$\frac{\partial(TC)}{\partial(y)} = \frac{h}{2} \frac{2(y - b)y - (y - b)^2}{y^2} - \frac{DK}{y^2} - \frac{b^2 p}{2y^2} = 0$$

$$= y^2 - \frac{2DK}{h} \left( \frac{h + p}{h} \right) b^2 = 0$$

Substituting $b^* = \frac{h}{h+p} y^*$ in the above equation gives

$$y^* = \sqrt{\frac{2DK}{h} \cdot \frac{h + p}{p}}$$

From equation (6), the interval between orders can be calculated as follows:

$$T^* = \frac{y^*}{D} \quad (8)$$

**Example:**

Suppose the North-west Electronics Company has a product for which the assumptions of the EOQ inventory model with backorders are valid. Information obtained by the company is as follows:

- The annual demand $D = 2000$ units/year
- The cost per order $K = £25$
- The holding cost rate $h = £10$ per unit per year
- The backorder cost $p = £30$ per unit per year

Work out the optimal order quantity $y^*$, the optimal backorder level $b^*$, and the cycle time $T$.

Solve:

The optimal order quantity

$$y^* = \sqrt{2 \times 2000 \times 25/10 \times (10+30)/30} = 115.47 \text{ units} \approx 115 \text{ units}$$

The optimal backorder level

$$b^* = \frac{10}{10+30} \times 115 = 28.75 \text{ units} \approx 29 \text{ units}$$

The cycle time for placing orders

$$T = \frac{y^*}{D} = \frac{115}{2000} = 0.0575 \text{ year}$$
Since there are 250 working days in a year, the cycle time is \(0.0575 \times 250 = 14.4\) working days.

The total annual cost for operating the inventory is

\[
TC = \text{holding cost} + \text{setup cost} + \text{shortage cost} \\
= 10 \times \frac{(115-29)^2}{2 \times 115} + 2000 \times \frac{25}{115} + 29^2 \times \frac{30}{2 \times 115} \\
= £322 + £435 + £110 = £867/\text{year}
\]

If the company had chosen to prohibit backorders and had adopted the regular EOQ model, the recommended inventory decision would have been

\[
y^* = \sqrt{2 \times 2000 \times 25/10} = 100 \text{ units}
\]

**Lecture 4**

2.4 Economic production-quantity model

In the models we discussed so far, we assumed that the orders are filled instantaneously. However, in some situations, that is not the case. Consider the following situation. When the inventory reaches its zero level, a production system starts its operation and add new items to the inventory while the inventory keeps on its normal operation. Of course, the production rate of items must be higher than that of the demand rate. This section discusses the model for this situation.

This inventory model is represented in Figure 6.

As in the EOQ model, we are now dealing with two costs, the holding cost and the setup cost. Namely
TC = holding cost + setup cost

**Holding cost**

We know that the holding cost equals to the product of the average inventory level and the holding cost per item per unit time. Let

\[
D = \text{the demand rate per unit time} \\
P = \text{the production rate per unit time} \\
h = \text{the holding cost per item per unit time}
\]

Then, according to Figure 6, \( T = y/D \), \( T_1 = y/P \), and

\[ T_2 = T - T_1 = \frac{y(P-D)}{DP} \]

The maximum inventory level in this situation is

the build-up speed \( \times T_1 = (P-D) \times y/P = y(P-D)/P \)

From the EOQ model, we know that the average inventory level is half of its maximum. In this case, it is \( \frac{1}{2} y(P-D)/P \). Therefore,

the holding cost = \( \frac{1}{2} y(P-D)/P \times h = \frac{1}{2} yh(P-D)/P \)

**Setup cost**

The setup cost is based on the number of production runs per unit time and the setup cost per run.

The number of production runs is \( D/y \). If the setup cost per run is \( K \), which includes the labour, material, and other relevant cost during the production, then the setup cost is

the setup cost = \( D/y \times K = DK/y \)

**Total cost**

The total cost for this model is

\[
TC = \frac{1}{2} yh(P-D)/P + DK/y \quad (9)
\]

The minimum-cost production quantity, often referred to as the economic production quantity, can be found by minimise \( TC \) with respect to \( y \), leading to

\[
y^* = \sqrt{\frac{2DK}{h} \cdot \frac{P}{P - D}} \quad (10)
\]

**Example**
Beauty Bar Soap is produced on a production line that has an annual capacity of 60,000 cases. The annual demand is estimated 26,000 cases, with the demand rate essentially constant throughout the year. The cleaning, preparation, and setup of the production line cost approximately $135. The manufacturing cost per case is $4.50, and the annual holding cost is figured at a 24% rate. Thus, the holding cost per case per year is $1.08. What is the recommended production quantity?

Solve:

We know that

\[
P = 60,000 \text{ cases/year} \\
D = 26,000 \text{ cases/year} \\
K = $135 \text{ per run} \\
h = $1.08 \text{ per case per year}
\]

So,

\[
y^* = \sqrt{\frac{2DK}{h}} \cdot \frac{P}{P - D} = \sqrt{\frac{2 \times 26000 \times 135}{1.08}} \cdot \frac{60000}{60000 - 26000} = 3387 \text{ cases}
\]

The total annual cost according to \( y^* \) is

\[
TC^* = \sqrt{2DKh} \cdot \frac{P - D}{P} = \sqrt{2 \times 26000 \times 135 \times 1.08} \cdot \frac{34000}{60000} = $2073
\]

The cycle time between production runs is

\[
T = \frac{y^*}{D} = \frac{3387}{26000} = 0.1327 \text{ year}
\]

Considering that there are 250 working days in a year,

\[
T = 250 \times 0.1327 \approx 33 \text{ working days}
\]

Thus, we should plan a production run of 3387 units about every 33 working days.

Lecture 5

3. Probabilistic Models

In practice, a large number of inventory situations cannot be described by the deterministic inventory models we discussed so far. In these cases, the demand is no longer constant and deterministic, but probabilistic instead. This type of demand is best described by the probability distribution.

3.1 Single-period inventory model with probabilistic demand
It is necessary to clarify the term “single period”. This term refers to the situation where the inventory will only be demanded in one time duration, and cannot be transferred to the next time duration. Newspaper selling is such an example. The newspaper ordered for today will not be sold tomorrow. Fashion selling is another example. Spring-summer designs will not sell during the autumn-winter season.

Increment analysis is a method that can be used to determine the optimal order quantity for a single-period inventory model. The increment analysis addresses the how-much-to-order question by comparing “the cost or loss of ordering one additional unit” with “the cost or loss of not ordering one additional unit”. Let

\[ c_o = \text{cost per unit of overestimating demand} \]
This cost represents the loss of ordering one additional unit which will not sell.

\[ c_u = \text{cost per unit of underestimating demand} \]
This cost represents the loss of not ordering one additional unit which could have been sold.

Suppose that the probability of the demand of the inventory items being more than a certain level \( y \) is \( P(D>y) \), and that the probability of the demand of the inventory items being less than or equal to this level \( y \) is \( P(D\leq y) \). Then, the expected loss (EL) will be either of the following:

For overestimation: \[ EL(y+1) = c_o P(D\leq y) \]
For underestimation: \[ EL(y) = c_u P(D>y) \]

The optimal value of the demand level, \( y^* \), being the optimal ordering quantity as well, can be found when

\[ EL(y^*+1) = EL(y^*) \tag{11} \]

This means

\[ c_o P(D\leq y^*) = c_u P(D>y^*) \tag{12} \]

From the study of Probability, it is known that

\[ P(D>y^*) = 1 - P(D\leq y^*) \tag{13} \]

Substituting (13) into (12), we have

\[ c_o P(D\leq y^*) = c_u [1 - P(D\leq y^*)] \]

Solving for \( P(D\leq y^*) \), we have

\[ P(D\leq y^*) = c_u / (c_u + c_o) \tag{14} \]
The above expression provides the general condition for the optimal order quantity $y^*$ in the single-period inventory model. The determination of $y^*$ depends on the probability distribution.

**Example1:** (uniform probability distribution)

Emma’s Shoe Shop is to order some new design men’s shoes for the next spring-summer season. The shoes cost £40 per pair and retail £60 per pair. If there are still shoes not sold by the end of July, they will be put on clearance sale in August at the price of £30 per pair. It is expected that all the remaining shoes can be sold during the sale.

For the size 10D shoes, it is found that the demand can be described by the uniform probability distribution, shown in Figure 7. The demand range is between 350 and 650 pairs, with average, or expected, demand of 500 pairs of shoes.

![Figure 7](image)

We are asked to help Emma’s Shoe Shop to determine the order quantity.

**Solve:**

It is essential to work out the overestimating cost $c_o$ and the underestimating cost $c_u$.

The cost per pair of overestimating demand is equal to the purchase cost minus the sale price per pair; that is

$$c_o = £40 - £30 = £10$$

The cost per pair of underestimating demand is the difference between the regular selling price and the purchase cost; that is

$$c_u = £60 - £40 = £20$$

Putting $c_o$ and $c_u$ into equation (14), we have

$$P(D \leq y^*) = c_u/(c_u+c_o) = 20/(20+10) = 2/3$$

We can find the optimal order quantity $y^*$ by referring to the assumed probability distribution shown in Figure 7 and finding the value of $y$ that will provide $P(D \leq y^*) = 2/3$. To do this, we note that in the uniform distribution the
probability is evenly distributed over the entire range of 350 - 650. Thus, we can satisfy the expression for $y^*$ by moving two-thirds of the way from 350 to 650. That gives

$$350 + 2/3 (650-350) = 550$$ pairs

namely, the optimal order quantity of size 10D shoes is 550 pairs.

**Example 2:** (Normal probability distribution)
Kremer Chemical Company has a contract with one of its customers to supply a unique liquid chemical product. Historically, the customer places order approximately every 6 months. Since the chemical needs 2 months ageing time, Kremer will have to make its production quantity decision before the customer places an order. Kremer’s inventory problem is to determine the number of kilograms of the chemical to produce in anticipation of the customer’s order.

Detailed information:
Kremer’s manufacturing cost: £15/kg
Fixed selling price: £20/kg
Underestimation (Kremer to buy substitute + transportation): £24/kg
Overestimation (Kremer to reprocess and sell the surplus) £5/kg

Normal probability distribution (Figure 8) best describes the possible demand.

\[
\text{Expected demand } \mu = 1000 \text{ kg} \\
\text{Standard deviation } \sigma = 100 \\
\text{1000 kg}
\]

Figure 8

Solve:
Underestimating cost: \( c_u = £24 - £20 = £4/kg \)
Overestimating cost: \( c_o = £15 - £5 = £10/kg \)
Therefore, \( P(D \leq y^*) = 4/(4+1) = 0.29 \)

In the table of “area of for the standard normal distribution”, all the area under the curve is 1, representing Probability = 1. However, only half of the area under the curve (probability) is concerned in this table because of the symmetry of the standard normal distribution curve. The area used to find \( z \) is
that between the vertical line through $z$ and the vertical line in the middle. For example, if this area is 0.3944, then the corresponding $z$ value is 1.25.

Now, for this problem, $P(D \leq y^*) = 0.29$. This is corresponding to about 1/3 of the area covered by the curve on the left. The area which is to be used to find $z$ is $0.5 - 0.29 = 0.21$.

This gives us $z = -0.55$ from the table. To go back to the de-standardised form, we have $z = (y^* - \mu)/\sigma$. Thus

$$y^* = \mu + \sigma z = 1000 - 100 \times 0.55 = 945 \text{ kg}$$

With the assumed normal probability distribution of demand, the Kremer Chemical Company should produce 945 kg of the chemical in anticipation of the customer’s order.

![Figure 9](image)

Note that in this case the cost of underestimation is less that that of overestimation. Thus, the Kremer Chemical Company is willing to risk a higher probability of under estimation and hence a higher probability of stockout. In fact, Kremer’s optimal order quantity has a 0.29 probability of having a surplus and a $1-0.29 = 0.71$ probability of a stockout.

3.2 An order-quantity, reorder-point inventory model with probabilistic demand

In the previous section, we considered a single-period inventory model with probabilistic demand. In this section, we will extend our discussion to a multi-period order-quantity, reorder-point model with probabilistic demand. This multi-period model has the following characteristics:

1) the inventory system operates continuously with many repeating periods or cycles;
2) inventory can be carried from one period to the next;
3) an order is placed whenever the inventory position reaches the reorder point;
4) since demand is probabilistic, the following cannot be determined in advance:
The inventory pattern can be described by the Figure 10.

The how-much-to-order decision

Although we are in a probabilistic demand situation, we can apply the EOQ model as an approximation of the best order quantity. That is

\[ y^* = \sqrt{\frac{2KD}{h}} \]

where, in this case, D is the expected annual demand.

The when-to-order decision

If the expected (or mean, average) demand is \( \mu \) per unit time, and the standard deviation is \( \sigma \), then the reorder point \( r \) can be expressed as

\[ r = \mu + z\sigma \quad (15) \]

where \( z \) is the number of standard deviation necessary to obtain the stockout probability, and it can be find from the standard normal probability distribution table according to the tolerance of stockout probability.

Example:

Dabco lighting distributors purchases a special high intensity light bulb for industrial lighting systems. Dabco would like a recommendation on how much to order and when to order so that a low-cost inventory policy can maintained.
The data:

\( \Rightarrow \) ordering cost is \( K = £12 \) per order;
\( \Rightarrow \) one bulb costs £6, and Dabco uses a 20% annual holding cost for its inventory \( (h = £6 \times 0.20 = £1.20) \)
\( \Rightarrow \) demand during 1 week lead time is best represented by a normal probability distribution with a mean of 154 light bulbs per week and a standard deviation of 25 light bulbs;
\( \Rightarrow \) Dabco is willing to tolerate an average of 1 stockout per year.

Solve:

**How-much-to-order**

Since the mean or expected demand is 154 bulbs, the annual demand is
\( 154 \times 52 \text{ weeks} = 8008 \text{ bulbs/year}. \)

\[ y^* = \frac{2KD}{h} = \sqrt{2 \times 12 \times 8008 / 1.20} \approx 400 \text{ bulbs} \]

Using \( y^* = 400 \), Dabco can anticipate placing approximately
\( D/y^* = 8008/400 \approx 20 \text{ orders per year} \)
with an average of approximately \( 250/20 = 12.5 \text{ working days between orders} \).

**When-to-order**

It is known that Dabco allows an average of 1 stockout per year. It has been calculated that there will be approximately 20 orders a year. So, the probability of a stockout in a year is 5% or 0.05. In other word, the probability for stockout not to happen is 0.5 - 0.05 = 0.45. This corresponds to \( z = 1.645 \).

Since \( \mu = 154 \) and \( \sigma = 25 \)
\[ r = \mu + z\sigma = 154 + 1.645 \times 25 \approx 195 \text{ bulbs} \]

Thus, the recommended inventory decision is to order 400 bulbs whenever the inventory level reaches the reorder point of 195.

Since the mean or expected demand during the lead time is 195 units, the 195-154 = 41 units serve as a safety stock, which absorbs higher-than-usual demand during the lead time. Roughly 95% of the time, the 195 units will be able to satisfy demand during the lead time.

The anticipated annual cost for this system is as follows:

<table>
<thead>
<tr>
<th>Description</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ordering cost</td>
<td>( DK/y^* = 8000 \times 12 / 400 = £240 )</td>
</tr>
<tr>
<td>Holding cost, normal inventory</td>
<td>( \frac{1}{2} y^* h = 0.5 \times 400 \times 1.20 = £240 )</td>
</tr>
<tr>
<td>Holding cost, safety stock</td>
<td>( (41)h = 41 \times 1.20 = £49.2 )</td>
</tr>
<tr>
<td>Total</td>
<td>£529.20</td>
</tr>
</tbody>
</table>
Note that if Dabco had a constant demand, the annual cost would be £480 only. The uncertainty in the situation costs an extra £49.20.

3.3 A periodic-review model with probabilistic demand

The inventory model discussed in 3.1 is a continuous-review model system, where the inventory position is monitored continuously so that an order can be placed whenever the reorder point is reached. With computerised inventory systems, such continuous-review on the inventory can be easily achieved.

However, if a company handles multiple products, continuous-review on each of the products may mean heavy work-load and probably low efficiency. In such cases, an alternative inventory model, the periodic review model, is preferred, because this model enables the orders for several items to be placed at the same preset periodic-review time.

In this model, we assume that for any single product, the lead time is less than the length of the review period. Then the how-much-to-order decision at any review period is determined using the following:

\[ y = M - H \]  \hspace{1cm} (16)

where, 
- \( y \) = the order quantity;
- \( M \) = the replenishment (or the maximum) level;
- \( H \) = the inventory on hand at the review period.

Since the demand is probabilistic, the inventory on hand, \( H \), will vary. Thus, the order quantity that must be sufficient to bring the inventory position back to its maximum or replenishment level \( M \) can be expected to vary each period. Under the periodic-review model, enough units are ordered each review period to bring the inventory position back to the replenishment level.

A typical inventory pattern for a periodic-review system with probabilistic demand is illustrated in Figure 11. Note that the time between periodic reviews is predetermined and fixed.
The decision variable in the periodic-review model is the replenishment level $M$. To determine $M$, we could begin by developing a total-cost model, including holding, ordering, and stockout (shortage) costs. Instead, we will describe an approach that is often used in practice. In this approach, the objective is to determine a replenishment level that will meet a desired performance level, such as a reasonably low probability of stockout or a reasonably low number of stockouts per year.

In the periodic-review model, the order quantity at each review period must be sufficient to cover demand for the review period plus the demand for the following lead time. If during the review period plus the lead-time period the demand can be expressed by the normal probability distribution, then the general expression for $M$ is

$$M = \mu + z\sigma$$  \hspace{1cm} (17)

where $\mu$ = the mean demand during the review period plus the lead-time period;
$\sigma$ = the standard deviation of demand;
$z$ = the number of standard deviations necessary to obtain the acceptable stockout probability.

**Example:**

Pound Discounts is a store selling a wide variety of products for household use. The company operates its inventory system with a two-week periodic review. From past experience, Pound discount knows that the demand for any kind of its goods obeys the normal probability distribution. For product, the mean demand during the 2-week review period and the lead-time period is 250 units with the standard deviation of demand being 45. If there is 1% chance that demand will exceed the replenishment level, work out the replenishment (or maximum) inventory level the store should keep for this product.

Solve:

Known that $\mu = 250$, $\sigma = 45$, and there is a 1% chance of stockout
From the normal probability distribution table and using 1% stockout probability, we know that
\[ z = 2.33 \]

Therefore,

\[ M = \mu + z\sigma = 250 + 2.33 \times 45 \approx 355 \text{ units} \]
Exercise 4

Inventory Models
(Stock Control)

1. In each of the following cases, inventory is replenished instantaneously and no shortage is allowed. Find the economic lot size, the associated total cost, and the length of time between two orders.
   a. $K = £100$ per order, $h = £0.05$ per unit per day, $D = 30$ units per day.
   b. $K = £50$ per order, $h = £0.05$ per unit per day, $D = 30$ units per day.
   c. $K = £100$ per order, $h = £0.01$ per unit per day, $D = 40$ units per day.
   d. $K = £100$ per order, $h = £0.04$ per unit per day, $D = 20$ units per day.

   The symbols used:
   - $K$ - the set-up cost per order
   - $h$ - the holding cost per item per unit time
   - $D$ - the demand rate in items per unit time

2. In each case in Problem 1, determine the reorder point assuming that the lead time is
   a. 14 days.
   b. 40 days.

3. A company currently purchases its stock of a certain item by ordering enough supply to cover a 1-month demand. The annual demand of the item is 1500 units. It is estimated that it costs £20 every time an order is placed. The holding cost per unit inventory per month is £2 and no shortage is allowed.
   a. Determine the optimal order quantity and the time between orders.
   b. Determine the difference in annual inventory costs between the optimal policy and the current policy of ordering 1-month supply 12 times a year.

4. An item is consumed at the rate of 30 items a day. The holding cost per item per unit time is £0.05 and the set-up cost is £100. Suppose that no shortage is allowed and the purchasing cost is £10 for any quantity less than or equal to $q = 300$ and £8 otherwise. Find the economic lot size. What is the answer if $q = 500$ instead?

5. Consider the inventory situation with the following features:
   - Single item
   - Constant demand rate
   - Instantaneous replenishment
   - No shortage allowed

   Notations to be used:
   - $C_o = $ the cost of placing one order, i.e. the set-up cost;
   - $C_h = $ the holding cost per item per unit time;
   - $Y = $ the order quantity; and
D = demand rate for an item.

a. Draw at least 4 cycles of the inventory variations against time, indicating the maximum inventory level and the cycle time.
b. Decide the expression for the total inventory cost.
c. Determine “how much to order” every time an order is placed.
d. Determine “when to order” in terms of inventory position R, assuming the lead time is L.

6. An item sells for £4 a unit but a 10% discount is offered for lots of size 150 units or more. A company that consumes this item at the rate of 20 items per day wants to decide whether or not to take advantage of the discount. The set-up cost for ordering a lot is £50 and the holding cost per unit per day is £0.30. Should the company take advantage of the discount?

7. The ABC Company purchases a component used in the manufacture of automobile engines directly from the supplier. ABC’s engine production requires 12,000 items of this component annually. The cost for placing an order is $25, and the holding of the component costs $0.50 per item per year. The Company decided to operate with a backorder (shortage) inventory policy. Backorder costs are estimated to be $5 per unit per year. Identify the following:
   a. The minimum-cost order quantity
   b. The maximum number of backorders
   c. The maximum inventory level
   d. The cycle time
   e. Total annual cost
   f. Assuming 250 days of operation per year and a lead time of 5 days, what is the reorder point for the ABC Company?

8. A manufacturing company operates an inventory with the economic production lot size model, where the production rate is 8000 units per year, the demand rate is 2000 units per year, the ordering cost is $300 for placing an order, and the holding cost is $1.60 per unit per year. Find the following:
   a. the economic production lot size for this situation;
   b. the minimum cost of keeping the inventory; and
   c. the production frequency.

9. Wilson Publish Company produces books for the retail market. Demand for a current book is expected to occur at a constant annual rate of 7200 copies. The cost of one copy of the book is £14.50. The holding cost is based on an 18% annual rate, and the production set-up costs are £150 per set-up. The equipment on which the book is produced has an annual production volume of 25,000 copies. There are 250 working days per year and the lead time for a production run is 15 days. Use the production lot size model to compute the following values:
   a. Minimum-cost production lot size
   b. Number of production runs per year
c. Cycle time

d. Length of production run

e. Maximum inventory level

f. Total annual cost

g. Reorder point

10. A retail outlet sells a seasonal product for $10 per unit. The cost of the product is $8 per unit. All units not sold during the regular season for half the retail price in an end-of-season clearance sale. Assume that the demand for the product is uniformly distributed between 200 and 800.

a. What is the recommended ordering quantity?

b. What is the probability of a stockout using your order quantity in (a)?

c. To keep customers happy and returning to the store later, the owner feels that stockouts should be avoided if at all possible. What is your recommended order quantity if the owner is willing to tolerate a 0.15 probability of stockout?

d. Using your answer to (c), what is the goodwill cost you are assigning to a stockout?

11. Floyd Distributors, Inc., provide a variety of auto parts to small local garages. Floyd purchases parts from manufacturers according to the EOQ model and ships the parts from a regional warehouse directly to its customers. For a particular type of muffler, Floyd’s EOQ analysis recommends orders with \( y^* = 25 \) to satisfy an annual demand of 200 mufflers. There are 250 working days per year and the lead time averages 15 days.

a. What is the reorder point if Floyd assumes a constant demand rate?

b. Suppose that an analysis of Floyd’s muffler demand shows that the demand follows a normal probability distribution with \( \mu = 12 \) and \( \sigma = 2.5 \). If Floyd management can tolerate one stockout per year, what is the revised reorder point?

c. What is the safety stock for (b)? If the holding cost is $5 per unit per year, what is the extra cost due to the uncertainty of demand?